ELSEVIER
Topology and its Applications 123 (2002) 421–427
TOPOLOGY AND ITS APPLICATIONS
www.elsevier.com/locate/topol

Inverse limits of Markov interval maps

Sarah E. Holte

Division of Public Health Sciences, Fred Hutchinson Cancer Research Center, Seattle, WA 98109-1024, USA

Received 1 June 2000; received in revised form 27 January 2001

Abstract

Inverse limit spaces of one-dimensional continua frequently appear as attractors in dissipative dynamical systems. As such, there has been considerable interest in the topology of these inverse limit spaces. In this work we describe the topology of Markov interval maps, and use our results to show that for unimodal interval maps with finite kneading sequences, the kneading sequence and dynamics of the left endpoint determine the topology of the associated inverse limit space.

Keywords: Inverse limit space; Markov map; Kneading sequence; Attractor

In recent years it has become apparent that inverse limits of one-dimensional continua play a role in the study of certain dissipative dynamical systems. For a general discussion of this relationship see [16]. In particular, inverse limits of interval maps where all the critical points are periodic arise as attractors of planar diffeomorphisms [1,5,6,10–14,17]. This phenomena has motivated the study of the topological structure of such inverse limit spaces. Recent work [2–4] describes topological properties of inverse limits of maps from the family of tent maps of the interval. The purpose of this work is to relate inverse limits of the interval with Markov bonding maps from different families of maps, e.g., tent maps and quadratic maps. We also obtain the result that a finite kneading sequence of a unimodal interval map and the dynamics of the left endpoint of the map, determine the topology of the associated inverse limit space.

Given an interval I and a surjective self-map, $f : I \to I$ we define the *the inverse limit space of f on I*, denoted as (I, f), as the topological subspace if I^∞ such that $(x_0, x_1, \ldots) \in (I, f)$ if and only if $x_i = f(x_{i+1})$ for all $i \geqslant 0$. A complete description of topological inverse limit spaces can be found in [15].

We say that a map $f : [a_0, a_m] \to [a_0, a_m]$ is *Markov* if there exists a finite set of points, $A = \{a_0 < \cdots < a_m\}$ such that $f(A) \subset A$ and the restriction of f to each

E-mail address: sarah@hivnet.fhcrc.org (S.E. Holte).

0166-8641/01/$ – see front matter
PII: S0166-8641(01)00209-7

component of $[a_0, a_m] - A$ is one-to-one. If $f:[a_0, a_m] \to [a_0, a_m]$ is Markov with respect to $A = \{a_0 < a_1 < \cdots < a_m\}$, $g:[b_0, b_m] \to [b_0, b_m]$ is Markov with respect to $B = \{b_0 < b_1 < \cdots < b_m\}$, and $f(a_j) = a_k$ if and only if $g(b_j) = b_k$, we say that f and g are Markov with same pattern. Theorem 1 describes a situation where Markov interval maps produce homeomorphic inverse limit spaces.

Theorem 1. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of surjective self-maps of $[a_0, a_m]$ which are all Markov with respect to $A = \{a_0, \ldots, a_m\}$ and $\{g_n\}_{n=1}^{\infty}$ be a sequence of surjective self-maps of $[b_0, b_m]$ which are all Markov with respect to $B = \{b_0, \ldots, b_m\}$. If $\{f_n, g_n\}_{n=1}^{\infty}$ are all Markov with the same pattern then (I, f_n) is homeomorphic to (I, g_n).*

Proof. Let I_j denote the interval $[a_j, a_{j+1}]$ and J_j denote the interval $[b_j, b_{j+1}]$. We will show that if $(x_0, x_1, \ldots) \in (I, f_n)$ then there exists a unique point $(y_0, y_1, \ldots) \in (I, g_n)$ such that $y_0 = x_0$ and $x_n \in I_j$ if and only if $y_n \in J_j$. Then we can define $\phi:(I, f_n) \to (I, g_n)$ by setting $\phi((x_0, x_1, \ldots))$ equal to the unique point of (I, g_n) described above. To complete the proof we will show that ϕ is one-to-one, onto, and continuous.

To define ϕ, let $(x_0, x_1, \ldots)$ be an element of (I, f_n). Let $h: I \to I$ be a homeomorphism with $h(a_j) = b_j$, $0 \leqslant j \leqslant m$. We inductively define a nested sequence $\{Q_n\}_{n=0}^{\infty}$ of closed, non-empty subsets of (I, g_n) with the following properties: if $(y_0, y_1, \ldots) \in Q_n$, then $y_0 = h(x_0)$, and $y_i \in J_j$ if and only if $x_i \in I_j$ for $0 \leqslant i \leqslant n$. Let $Q_0 = \pi_0^{-1}(h(x_0)) \subset (I, g_n)$. Then Q_0 is closed and non-empty. Now suppose $Q_n \subset Q_{n-1} \subset \cdots \subset Q_0$ satisfy the above properties. Define Q_{n+1} as follows: let $(y_0, y_1, \ldots)$ be an element of Q_n, $J_{j(n)}$ an interval which contains y_n, $I_{j(n)}$ the corresponding interval which contains x_n, and $I_{j(n+1)}$ an interval which contains x_{n+1}. Then $f_n(x_{n+1}) = x_n \in I_{j(n)}$ so that $f_n(I_{j(n+1)}) \cap I_{j(n)} \neq \emptyset$. Also, f_n is invariant on A, so $f_n(I_{j(n+1)}) = [a_{k_1}, a_{k_2}]$, where a_{k_1} and a_{k_2} are elements of A. Thus $I_{j(n)} \subset f_n(I_{j(n+1)})$ or $I_{j(n)} \cap f_n(I_{j(n+1)}) = \{x_n\}$. If $I_{j(n)} \subset f_n(I_{j(n+1)})$ then $J_{j(n)} \subset g_n(J_{j(n+1)})$ and so $y_n \in J_{j(n)} \subset g_n(J_{j(n+1)})$. Thus, there exists $y_{n+1} \in J_{j(n+1)}$ such that $g_n(y_{n+1}) = y_n$. In this case, set $Q_{n+1} = \pi_{n+1}^{-1}(y_{n+1}) \subset (I, g_n)$. If $I_{j(n)} \cap f_n(I_{j(n+1)}) = \{x_n\}$, then $x_n = a_{j(n)}$ or $x_n = a_{j(n)+1}$, and $x_{n+1} \in A$, say $x_{n+1} = a_k$. Note that $f_n(a_k) = x_n$. Then $y_n = b_{j(n)}$ or $y_n = b_{j(n)+1}$. In this case, let $y_{n+1} = b_k$. Since f_{n+1} and g_{n+1} are Markov with the same pattern, it follows that $g_{n+1}(y_{n+1}) = y_n$. Let $Q_{n+1} = \pi_{n+1}^{-1}(y_{n+1}) \subset (I, g_n)$. Obviously Q_{n+1} is closed and non-empty and it is easy to check that $Q_{n+1} \subset Q_n$.

Since each Q_n is closed and non-empty, and the sets $Q_0, Q_1, \ldots$ are nested, it follows that there exists $(y_0, y_1, \ldots) \in \bigcap Q_n$. Suppose that $(y'_0, y'_1, \ldots)$ is another point of $\bigcap Q_n$. Let i be the first coordinate so that $y'_i \neq y_i$. Then $i > 0$ since $y'_0 = y_0 = h(x_0)$. Also, y_i and y'_i are both elements of some interval J_j and $g_{i-1}(y_i) = g_{i-1}(y'_i) = y_{i-1}$. But this contradicts the fact that g_{i-1} is one-to-one on J_j. Therefore there is only one point in $\bigcap Q_n$. Thus we define $\phi:(I, f_n) \to (I, g_n)$ by setting $\phi((x_0, x_1, \ldots))$ equal to the unique point $(y_0, y_1, \ldots) \in (I, g_n)$ such that $h(x_0) = y_0$ and $x_n \in I_j$ if and only if $y_n \in J_j$.

The same construction shows that given a point $(y_0, y_1, \ldots) \in (I, g_n)$ we can find a unique point $(x_0, x_1, \ldots) \in (I, f_n)$ such that $h(x_0) = y_0$, and $x_n \in I_j$ if and only if $y_n \in J_j$. It follows that ϕ is one-to-one and onto. We now show that ϕ is continuous.

Let $(x_0, x_1, \ldots) \in (I, f_n)$, $(y_0, y_1, \ldots) = \phi((x_0, x_1, \ldots))$, and L be the minimum of the lengths of the intervals $[a_j, a_{j+1}]$. Given $\varepsilon > 0$, choose N so that $\sum_{n=N}^{\infty} \frac{1}{2^n} < \frac{1}{2}\varepsilon$. For each $n \in \mathcal{N}$ and j between 0 and m, $g_n^{-1} : g_n(J_j) \to J_j$ is a homeomorphism since $g_n|J_j$ is one-to-one. From now on, let g_n^j denote $g_n^{-1} : g_n(J_j) \to J_j$. Note that if $y_{n+1} \in J_j$, then $y_{n+1} = g_n^j(y_n)$. Next, for each i, $0 \leqslant i \leqslant N+1$, define L_i as follows: if $x_i \in A$, let $L_i = L$. If $x_i \notin A$, then $x_i \in (a_j, a_{j+1})$. In this case, let $L_i = \min\{a_{j+1} - x_i, x_i - a_j\}$. Note that if $|x_i - x_i'| < L_i$ then x_i and x_i' both lie in some I_j.

Next, as we noted above, $g_{N-1}^j : g_{N-1}(J_j) \to J_j$ is a homeomorphism. Thus, for each j, $1 \leqslant j \leqslant m$, we may choose δ_{N-1}^j so that if y and y' are elements of $g_{N-1}(J_j)$ with $|y - y'| < \delta_{N-1}^j$, then

$$\left|g_{N-1}^j(y) - g_{N-1}^j(y')\right| < \frac{1}{2N}\,\varepsilon.$$

Let $\delta_{N-1} = \min\{\delta_{N-1}^1, \ldots, \delta_{N-1}^m\}$. Similarly, for each j, $1 \leqslant j \leqslant m-1$, choose δ_{N-2}^j so that if y and y' are elements of $g_{N-2}(J_j)$ with $|y - y'| < \delta_{N-2}^j$, then $|g_{N-2}^j(y) - g_{N-2}^j(y')| < \min\{\frac{1}{2N}\varepsilon, \delta_{N-1}\}$. Let $\delta_{N-2} = \min\{\delta_{N-2}^1, \ldots, \delta_{N-2}^m\}$.

Continue in this way to choose $\delta_{N-(i+1)}^j$ so that if y and y' are elements of $g_{N-(i+1)}(J_j)$ with $|y - y'| < \delta_{N-(i+1)}^j$, then

$$\left|g_{N-(i+1)}^j(y) - g_{N-(i+1)}^j(y')\right| < \min\left\{\frac{1}{2N}\varepsilon, \delta_{N-i}\right\}.$$

Let $\delta_{N-(i+1)} = \min\{\delta_{N-(i+1)}^1, \ldots, \delta_{N-(i+1)}^m\}$. Thus we obtain $\delta_0, \delta_1, \ldots, \delta_{N-1}$ such that if y and y' are elements of $g_i(J_j)$ with $|y - y'| < \delta_i$, then $|g_i^j(y) - g_i^j(y')| < \min\{\delta_{i+1}, \frac{1}{2N}\varepsilon\}$.

Finally, choose δ_{-1} so that if $|x - x'| < \delta_{-1}$, then $|h(x) - h(x')| < \min\{\delta_0 \frac{1}{2N}\varepsilon\}$. Let $\delta = \min\{\delta_{-1}, L_0, \frac{L_1}{2}, \ldots, \frac{L_{N+1}}{2^{N+1}}, \frac{1}{2N}\varepsilon\}$. Now suppose that $(x_0', x_1', \ldots) \in (I, f_n)$ such that $d((x_0, x_1, \ldots), (x_0', x_1' \ldots)) < \delta$. Let $(y_0', y_1', \ldots)$ denote $\phi((x_0', x_1' \ldots))$. Then for $0 \leqslant i \leqslant N+1$, it follows that $|x_i - x_i'| < L_i$. Therefore, there exists $I_{j(i)}$ such that x_i and x_i' are both elements of $I_{j(i)}$. This implies that y_i and y_i' are both elements of $J_{j(i)}$.

We now show inductively that $|y_i - y_i'| < \min\{\delta_i, \frac{1}{2N}\varepsilon\}$ for $0 \leqslant i \leqslant N$. Since $|x_0 - x_0'| < \delta \leqslant \delta_{-1}$, it follows that $|y_0 - y_0'| < \min\{\delta_0, \frac{1}{2N}\varepsilon\}$. Now suppose that $|y_i - y_i'| < \min\{\delta_i, \frac{1}{2N}\varepsilon\}$. Let $J_{j(i+1)}$ be an interval which contains y_{i+1} and y_{i+1}'. Then y_i and y_i' are elements of $g_i(J_{j(i+1)})$. Furthermore, $|y_i - y_i'| < \delta_i$ by the induction hypothesis, so $|g_i^j(y_i) - g_i^j(y_i')| < \min\{\delta_{i+1}, \frac{1}{2N}\varepsilon\}$. But $g_i^j(y_i) = y_{i+1}$ and $g_i^j(y_i') = y_{i+1}'$. Therefore $|y_{i+1} - y_{i+1}'| < \min\{\delta_{i+1}, \frac{1}{2N}\varepsilon\}$. It follows that

$$\begin{aligned} d\big((y_0, y_1, \ldots), (y_0', y_1', \ldots)\big) &= \sum_{i=0}^{\infty} \frac{|y_i - y_i'|}{2^i} = \sum_{i=0}^{N-1} \frac{|y_i - y_i'|}{2^i} + \sum_{i=N}^{\infty} \frac{|y_i - y_i'|}{2^i} \\ &\leqslant \sum_{i=0}^{N-1} \frac{1}{2N}\,\varepsilon + \frac{1}{2}\,\varepsilon = \varepsilon. \end{aligned}$$

Thus ϕ is continuous, and this completes the proof of the theorem.

We now turn our attention to interval maps, f, which are unimodal, i.e., there exists a unique point c in the interior of I such that f is strictly increasing on $\{x\colon\ x < c\}$ and strictly decreasing on $\{x\colon\ x > c\}$. We use the definitions and theorems on kneading theory of unimodal maps found in [8]. Specifically, if f is a unimodal interval map the itinerary of point x under f is an infinite or finite sequence, $S(x) = (s_0, s_1, \ldots)$, in the symbols R, L, and C where

$$s_j = \begin{cases} R & \text{if } f^j(x) > c, \\ L & \text{if } f^j(x) < c, \\ C & \text{if } f^j(x) = c \end{cases}$$

and the sequence terminates the first time the symbol C is realized in the sequence, i.e., if $f^j(x) = c$ for some $j \geqslant 1$ and $f^l(x) \neq c$ for all $l < j$. The kneading sequence of f, denoted by $K(f)$, is the itinerary of $f(c)$. We will also utilize the order on itineraries described in [8, pp. 65–66].

As a corollary to Theorem 1, we obtain a result that shows how finite kneading sequences determine the topology of inverse limits of an interval with a unimodal bonding map.

Corollary 1. *Let f be a surjective unimodal self-map of $I = [a_0, a_m]$ and g be a surjective unimodal self-map of $J = [b_0, b_m]$ with the same finite kneading sequence as f. Furthermore, assume that $f(a_0) = a_0$ if and only if $g(b_0) = b_0$. Then (I, f) is homeomorphic at (J, g).*

It is sometimes the case (see, for example, [9, p. 137]) that the definition of unimodal maps includes the requirement that the left endpoint of the interval be fixed by the map. If this definition is used, then the second condition of the corollary obviously holds for any two unimodal maps and so may be eliminated from the statement of the theorem.

Let $\mathcal{O}(x)$ denote the orbit of the point x. To prove the corollary, let $A = \mathcal{O}(c_f) \cup \{a_0\}$ and let $B = \mathcal{O}(c_g) \cup \{b_0\}$. Note that the critical point of f, c_f is an element of A and that the critical point of g, c_g, is an element of B. Furthermore, $a_m \in A$ since f is surjective so that $f(c_f) = a_m$. Similarly $b_m \in B$. Since c_f and c_f are periodic with the same period, A and B are finite sets of the same size, $A = \{a_0 < a_1 < \cdots < a_m\}$ and $B = \{b_0 < b_1 < \cdots < b_m\}$. To see that $f(A) \subset A$ note that if $f(a_0) \neq a_0$, then $f(a_m) = a_0$ since f is surjective. This implies that $a_0 \in \mathcal{O}(c_g)$ since $a_m \in \mathcal{O}(c_f)$ and so in this case, $A = \mathcal{O}(c_f)$. Similarly $g(B) \subset B$. It follows that f is Markov with respect to A, and g is Markov with respect to B, so to prove the corollary we must show that $f(a_i) = a_j$ if and only if $g(b_i) = b_j$.

First suppose that $f(a_0) = a_0$ and $g(b_0) = b_0$. Then $\mathcal{O}(c_f) = \{a_1 < \cdots < a_m\}$ and $\mathcal{O}(c_g) = \{b_1 < \cdots < b_m\}$. Since c_f and c_g have the same finite itinerary, we have the following relationship between itineraries of points in $\mathcal{O}(c_f)$ and $\mathcal{O}(c_g)$: $S(a_1) = S(b_1) < S(a_2) = S(b_2) < \cdots < S(a_m) = S(b_m)$. This follows from [8, Lemma II.1.2] and the fact that c_f and c_g have the same finite itinerary. Now suppose that $S(a_i) = (s_0, s_1, \ldots, C)$ and $f(a_i) = a_j$. Then $S(b_i) = (s_0, s_1, \ldots, C)$ and $S(a_j) = S(f(a_i)) = (s_1, \ldots, C)$. Also, $S(g(b_i)) = (s_1, \ldots, C) = S(b_j)$. Since each element of $\mathcal{O}(c_g)$ has a unique itinerary, it must be the case that $g(b_i) = b_j$. Therefore $f(a_i) = a_j$ if and only if $g(b_i) = b_j$ and so f and g satisfy the conditions of Theorem 1, so that (I, f) is homeomorphic to (J, g).

Next suppose that $f(a_0) \neq a_0$. Then $g(b_0) \neq b_0$. Then, as noted above, $A = \mathcal{O}(c_f)$ and $B = \mathcal{O}(c_g)$. The argument given above using itineraries of points in A and B proves that $f(a_i) = a_j$ if and only if $g(b_i) = b_j$, and so (I, f) is homeomorphic to (J, g) by Theorem 1.

In the remainder of this article, we address the situation where an interval map is Markov in the sense that its critical points are periodic, but the map is monotone rather than one-to-one on the components of the complement of the orbits of critical points. Such maps include maps in the trapezoid family of interval maps where the plateau is periodic, as well as interval maps which define the inverse limits which are the full attracting sets of the planar diffeomorphisms described in [1,10,11]. These situations lead to the following definitions.

Suppose that $A = \{a_0 < \cdots < a_m\} \subset I = [a_0, a_m]$ and $f : I \to I$ such that the following conditions hold:

(i) $f(A) \subset A$,

(ii) $f|_{[a_i, a_{i+1}]}$ is monotone for each $i = 0, \ldots, m-1$,

(iii) there exist a finite number of (possibly degenerate) closed subintervals $C_0, \ldots, C_r$ of I such that f is one-to-one on each component of $I - \bigcup_{i=0}^{r} C_i$, $A \subset \bigcup_{i=0}^{r} C_i$, and f is constant on C_i for $i = 0, \ldots, r$.

In this case we say that f is *m-Markov* (*monotone Markov*) *with respect to* A. If $f : [a_0, a_m] \to [a_0, a_m]$ is m-Markov with respect to $A = \{a_0 < a_1 < \cdots < a_m\}$, $g : [b_0, b_m] \to [b_0, b_m]$ is m-Markov with respect to $B = \{b_0 < b_1 < \cdots < b_m\}$, and $f(a_j) = a_k$ if and only if $g(b_j) = b_k$ we say that f and g are m-Markov with the same pattern.

Next suppose that $f : I \to I$, C is a subinterval of the interior of I, and R and L are the components of $I - C$ such that if $x \in R$, then $x > y$ for all $y \in C$. If f is strictly increasing on L, strictly decreasing on R, and constant on C, then we say that f is *m-unimodal* (*monotone unimodal*). In this case, let $M = f(C)$. We are interested in m-unimodal maps where the plateau, M, is periodic. We define itineraries of points in the following way: Since M is periodic, there exists n such that $f^n(M) = M$. Let $c = f^{n-1}(M)$. Then $c \in C$ and we define the m-itinerary of a point x to be an infinite or finite sequence, $S(x) = (s_0, s_1, \ldots)$, in the symbols R, L, and C where

$$s_j = \begin{cases} R & \text{if } f^j(x) > c, \\ L & \text{if } f^j(x) < c, \\ C & \text{if } f^j(x) = c \end{cases}$$

and the sequence terminates the first time the symbol C is realized in the sequence, i.e., if $f^j(x) = c$ for some $j \geqslant 1$ and $f^l(x) \neq c$ for all $l < j$. The m-kneading sequence of f, denoted by $K(f)$, is the m-itinerary of $M = f(c)$.

Theorem 2 describes a situation where m-Markov interval maps produce homeomorphic inverse limit spaces and Corollary 2 generalizes Corollary 1 for m-unimodal interval maps.

Theorem 2. *Suppose that f and g are surjective self-maps of $[a_0, a_m]$, f is m-Markov with respect to $A = \{a_0 < a_1 < \cdots < a_m\}$ and g is Markov with respect to A and has the* *same pattern as f. Then (I, f) is homeomorphic to (I, g). It follows that any two Markov or m-Markov interval maps which have the same pattern produce homeomorphic inverse limit spaces.*

Proof. Since f is m-Markov, there exist a finite number of subintervals $C_0, \dots, C_r$ such that f is one-to-one on each component of $[a_0, a_m] - \bigcup_{i=0}^{r} C_i$ and f is constant on C_i for $i = 0, \dots, r$. Let $D = [a_0, a_m] - \bigcup_{i=1}^{r} C_i$ and $D_1 < D_2 < \cdots < D_r$ be the components of D. Note that f is one-to-one on D_i for $i = 1, \dots, r$ and each D_i is an interval (b_i, c_i). Let $c_0 = a_0$ and $b_{m+1} = a_m$. For each $i = 1, \dots, r$, let $\{b_i^n\}_{n=1}^{\infty}$ be a strictly decreasing sequence of converging to b_i and $\{c_i^n\}_{n=1}^{\infty}$ be a strictly increasing sequence converging to c_i such that $b_i^1 < c_i^1$.

Define f_n as follows: $f_n(x) = f(x)$ if $x \in A \cup \bigcup_{i=1}^{r} (b_i^n, c_i^n)$. For each $i = 0, \dots, r$, if $C_i \cap A = \{a_j\}$, define f_n to be linear on (c_i^n, a_j) and (a_j, b_{i+1}^n). If $C_i \cap A = \emptyset$, define f_n to be linear on (c_i^n, b_{i+1}^n). It is straightforward to check that $f_n \to f$ uniformly as $n \to \infty$. Thus it follows from Theorem 3 of [7] that (I, f) is homeomorphic to (I, f_{n_k}) where $\{f_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{f_n\}_{n=1}^{\infty}$. In addition, each f_{n_k} is Markov with respect to A with the same pattern as f and g. Thus, by Theorem 1, (I, f_{n_k}) is homeomorphic to (I, g) and so (I, f) is homeomorphic to (I, g). □

Corollary 2. *Let $f : [a_0, a_m] \to [a_0, a_m]$ be a m-unimodal interval map with a periodic plateau and let $g : [b_0, b_m] \to [b_0, b_m]$ be a unimodal interval map with the same finite kneading sequence as the m-kneading sequence of f and such that $f(a_0) = a_0$ if and only if $g(b_0) = b_0$. Then $([a_0, a_m], f)$ is homeomorphic to $([b_0, b_m], g)$. It follows that any two unimodal or m-unimodal interval maps which have the same finite kneading or m-kneading sequence and same dynamic behavior of their left endpoints produce homeomorphic inverse limit spaces.*

Proof. Let $A = \mathcal{O}(M) \cup \{a_0\}$. Then f is m-Markov with respect to A. Let c denote the critical point of g and $B = \mathcal{O}(g(c)) \cup \{b_0\}$. Then g is Markov with respect to B and has the same pattern as f since the kneading sequence of g is the same as the m-kneading sequence of f and $f(a_0) = a_0$ if and only if $g(b_0) = b_0$. Thus $([a_0, a_m], f)$ is homeomorphic to $([b_0, b_m], g)$ by Theorem 2. □

References

[1] M. Barge, Horseshoe maps and inverse limits, Pacific J. Math. 121 (1986) 29–39.
[2] M. Barge, K. Brucks, B. Diamond, Self-similarity in inverse limit spaces of the tent family, Proc. Amer. Math. Soc. 124 (1996) 3563–3570.
[3] M. Barge, B. Diamond, Homeomorphisms of inverse limit spaces of one-dimensional maps, Fund. Math. 146 (1995) 171–187.
[4] M. Barge, B. Diamond, Inverse limits space of infinitely renormalizable maps, Topology Appl. 83 (1998) 103–108.
[5] M. Barge, S. Holte, Nearly one-dimensional Henon attractors and inverse limits, Nonlinearity 8 (1995) 29–42.

[6] L. Block, Diffeomorphisms obtained from endomorphisms, Trans. Amer. Math. Soc. 214 (1979) 403–413.
[7] M. Brown, Some applications of an approximation theorem for inverse limits, Proc. Amer. Math. Soc. 11 (1960) 478–483.
[8] P. Collet, J.-P. Eckmann, Iterated Maps on the Interval as Dynamical Systems, Birkhäuser, Basel, 1980.
[9] R.L. Devaney, An Introduction to Chaotic Dynamical Systems, 2nd Edition, Addison-Wesley, Reading, MA, 1989.
[10] S. Holte, Generalized horseshoe maps and inverse limits, Pacific J. Math. 156 (1992) 297–305.
[11] S. Holte, Embedding inverse limits of nearly Markov interval map as attractors in two dimensions, Colloq. Math. 68 (1995) 291–296.
[12] S. Holte, Full attracting sets of annulus maps which are inverse limits of circles, Topology Appl. 65 (1995) 49–56.
[13] S. Holte, R. Roe, Inverse limits associated with the forced van der Pol equation, J. Differential Equations Dynamical Systems 6 (1994) 601–612.
[14] M. Misiurewicz, Embedding inverse limits of interval maps as attractors, Fund. Math. 125 (1985) 23–40.
[15] S. Nadler, Continuum Theory: An Introduction, Marcel Dekker, New York, 1992.
[16] R. Schori, Chaos: an introduction to some topological aspects, in: M. Brown (Ed.), Continuum Theory and Dynamical Systems, American Mathematical Society, Providence, RI, 1991, pp. 149–161.
[17] W. Szczechla, Inverse limits of certain interval mappings as attractor in two dimensions, Fund. Math. 133 (1989) 1–23.